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WHEN SETS CAN OR CANNOT BE PRODUCT-DOMINANT

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Abstract. Given a finite set $A \subset \mathbb{R}$, we define

$$\begin{aligned} A+A &= \{a+a' \mid a,a' \in A\}, \\ A-A &= \{a-a' \mid a,a' \in A\}, \\ A.A &= \{aa' \mid a,a' \in A\}, \\ A/A &= \{a/a' \mid a,a' \in A,\ a' \neq 0\}. \end{aligned}$$

A set A is said to be sum-dominant or MSTD (More Sums than Differences) if $|A+A| > |A-A|$ and a set $A \subset \mathbb{R} \setminus \{0\}$ is said to be product-dominant or MPTQ (More Products than Quotients) if $|A.A| > |A/A|$. In this paper, we shall discuss several properties of MPTQ sets, investigate techniques of generating an infinite family of MPTQ sets, and identify some characterizations under which a finite set of numbers can or cannot be product-dominant. We confirm the existence of MPTQ sets of perfect squares and justify that n^{th} powers of prime numbers do not contain any MPTQ set. We extend the notion of MPTQ sets to the multiplicative group $\mathbb{Z}_p^*$ and recognize their correspondence with the MSTD sets in $\mathbb{Z}_{p-1}$.

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1. Introduction

Definition 1.1. *Given a finite set $A \subset \mathbb{R}$, we define*

$$\begin{aligned} A+A &= \{a+a' \mid a,a' \in A\}, \\ A-A &= \{a-a' \mid a,a' \in A\}, \\ A.A &= \{aa' \mid a,a' \in A\}, \\ A/A &= \{a/a' \mid a,a' \in A,\ a' \neq 0\}. \end{aligned}$$

Since addition is commutative but subtraction is not, we usually expect $|A-A| \geq |A+A|$. The conjecture wrongly attributed to J. H. Conway, says that there are no finite sets of integers with $|A+A| > |A-A|$. But Conway is said to have found the first example of a set $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$ in the 1960s, for which $|A+A| = 26 > 25 = |A-A|$.

A set A is said to be MSTD (More Sums than Differences) if $|A+A| > |A-A|$ and MDTS (More Differences than Sums) if $|A+A| < |A-A|$ [15]. In a similar way, a set $A \subset \mathbb{R} \setminus \{0\}$ is said to be MPTQ (More Products than Quotients) if $|A.A| > |A/A|$ and MQTP (More Quotients than Products) if $|A.A| < |A/A|$ [3]. In either case, we say that A is *balanced* if the cardinalities are equal.

The focus of research in Additive Number Theory has predominantly been on subsets of integers when it comes to MSTD sets. One may refer [6, 10, 13, 18, 19] for history and overview and [5, 7, 11, 12, 20] for explicit constructions. After Nathanson's review of the subject [15], there has been some attention given to the extension of the concept to finite groups and other settings [1, 8, 9, 21].

Martin and O'Bryant [11] demonstrated that as $n \to \infty$, the percentage of MSTD subsets in $\{1, 2, \dots, n\}$ remains above a positive constant. This breakthrough has led to significant advancements in the field of sum-dominant sets. Though the number of MSTD subsets of $[1, n]$ grows quite quickly as n grows, it is challenging to find MPTQ subsets of $[1, n]$. In fact, the set $[1, 36]$ does not contain any MPTQ subset. It remains a fact that MPTQ sets have not been studied widely compared to MSTD sets. Exploring the concept of MPTQ sets can yield valuable insights and uncover new findings about MSTD sets.

In this paper, we shall discuss various properties of MPTQ sets, give criteria for generating an infinite family of MPTQ sets, and identify several sets that can or cannot be MPTQ. We extend the notion of MPTQ sets to the multiplicative group $\mathbb{Z}_p^*$ for a prime p and recognize their correspondence with the MSTD sets in $\mathbb{Z}_{p-1}$ for a prime p. We also justify that $\mathbb{Z}_{\phi(n)}$ contains an MSTD set of cardinality $\phi(n)/2$ for each $n \in \mathbb{N}$, with $\phi(n) \geq 12$, of the form p^k or $2p^k$, where p is an odd prime and $k \in \mathbb{N}$.

2. Notations

The following notations will be used throughout.

- We write $b \to A$ to mean the adjoining of the number b to the set A that yields the set $A \cup \{b\}$ provided $b \notin A$ and we denote $A \cup \{b\}$ by A'.
- Given a set A of positive real numbers and $r \neq 0$ define $A^r = \{a^r : \ a \in A\}$.
- Given a set B of real numbers and $1 \neq r > 0$, define $r^B = \{r^b \mid b \in B\}$ and we call it the $r -$ *exponential transformation* of B.
- Given a set A of positive real numbers and $1 \neq r > 0$, define $\log_r A = \{\log_r a \mid a \in A\}$ and we call it the $r -$ *log transformation* of A.
- Given a set $A \subseteq \mathbb{R}$ and numbers λ, μ with $\lambda \neq 0$, we define the *dilation* of A to be $\lambda{*}A = \{\lambda a : a \in A\}$ and *translation* of A to be $A{+}\mu = \{a{+}\mu : a \in A\}$.
- $\triangle A = |A/A| - |A.A|$.
- For $n \in \mathbb{N}$ and $r \in \mathbb{R} \setminus \{0, \pm 1\}$, we denote by $G(r, n)$, the geometric progression $\{1, r, r^2, \dots, r^{n-1}\}$.
- We write MPTQ($\mathbb{Z}_p^*$) and MQTP($\mathbb{Z}_p^*$) for the collection of product-dominant and quotient-dominant sets in $\mathbb{Z}_p^*$ respectively.

3. Sets that are not Product-dominant

A property of a set is an *affine-invariant* if it remains unchanged under a dilation followed by a translation. The property of being an MSTD set is affine-invariant [15]. However, the same is not true for MPTQ sets. In fact, given a set $A \subseteq \mathbb{R}$ and a number λ with $\lambda \neq 0$,

$$|(\lambda * A).(\lambda * A)| = |\lambda^2 * (A.A)| = |A.A|, \text{ and } |(\lambda * A)/(\lambda * A)| = |A/A|.$$

i.e. the cardinalities of the product and quotient sets of A are invariant under dilation. However, in general, the cardinalities are not preserved under translation. Therefore the property of being an MPTQ set is not affine-invariant.

Example 3.1. *The set* $A = \{1, 2^2, 2^3, 2^4, 2^7, 2^{11}, 2^{12}, 2^{14}\}$ *is an MPTQ set, but* $B = A{+}1 = \{2, 5, 9, 17, 129, 2049, 4097, 16385\}$ *is not, as* $|B.B| = 36 < 57 = |B/B|$.

Let $A = \{a_1, a_2, \dots, a_n\}$ be a set of positive real numbers with $a_1 < a_2 < \dots < a_n$. We choose a real number $b > \dfrac{a_n^2}{a_1}$. With $b \to A$, we get $n+1$ new products,

$$ba_1 < ba_2 < \dots < ba_n < b^2,$$

and $2n$ new quotients,

$$\frac{b}{a_1} > \frac{b}{a_2} > \dots > \frac{b}{a_{n-1}} > \frac{b}{a_n},$$

along with their reciprocals.

Similarly, if we choose a number $0 < b < \dfrac{a_1^2}{a_n}$ then with $b \to A$, we get $n+1$ new products,

$$b^2 < ba_1 < ba_2 < \dots < ba_n,$$

and $2n$ new quotients,

$$\frac{b}{a_n} < \frac{b}{a_{n-1}} < \dots < \frac{b}{a_2} < \frac{b}{a_1},$$

along with their reciprocals. Thus, in either case, we get

$$\triangle A' = \triangle A + n - 1. \tag{1}$$

So $b \to A$ does not yield an MPTQ set unless $\triangle A \leq -n$. Therefore, in order to construct an MPTQ set by $b \to A$, where A is a set of positive reals with $\triangle A > -n$, the number b must be chosen such that

$$\frac{(\min A)^2}{\max A} \leq b \leq \frac{(\max A)^2}{\min A}$$

or equivalently,

$$(\min A)^3 \leq b\,(\min A)\,(\max A) \leq (\max A)^3.$$

Definition 3.2. [3] *A set A is said to be symmetric with respect to b if there exists $b \in \mathbb{R} \setminus \{0\}$ such that $A = b/A$.*

Since $|A.A| = |A.(b/A)| = |A/A|$, a symmetric set is always balanced.

Remark 3.3. *Let $A = \{a_1, a_2, \dots, a_n\}$ be a symmetric set of positive integers with $a_1 < a_2 < \dots < a_n$. If $b > 1$ is an integer having a prime factor p with $(p, a_i) = 1$ for each $i, 1 \leq i \leq n$, then $b \to A$ yields $n+1$ new products, and $2n$ new quotients. So A' is not MPTQ.*
Thus, to construct an MPTQ set of positive integers by $b \to A$, the number b must be chosen such that $\lceil \frac{a_1^2}{a_n} \rceil \leq b \leq \lfloor \frac{a_n^2}{a_1} \rfloor$, and it does not have any prime factor other than those appearing in the factorization of a_i's, $1 \leq i \leq n$.

Most of the examples of MPTQ sets listed in [3] are obtained by $b \to A$, where b is a suitable positive integer and $A \subset \{2^m 3^n : m, n \geq 0\}$ is symmetric. From above observations it follows that b can only be chosen among the numbers of the form $2^r 3^s$, with $r, s \geq 0,\ 2a - x \leq r \leq 2x - a,\ 2b - y \leq s \leq 2y - b$, where $2^x 3^y = \max A$ and $2^a 3^b = \min A$.

Example 3.4. $A = \{4, 9, 16, 18, 24, 162, 216, 243, 432, 972\} \subset \{2^m 3^n : m, n \geq 0\}$ *is symmetric with respect to* $2^4 3^5 = 3888$*, and by* $36 \to A$*, the resulting set* A' *is an MPTQ set with* $|A.A| = 52 > 51 = |A/A|$.

But there are also symmetric subsets of $\{2^m 3^n : m, n \geq 0\}$ for which $b \to A$ does not result in an MPTQ set for any choice of the positive integer b.

Example 3.5. *$A = \{1, 2, 4, 6, 9, 18, 36\}$ is symmetric with respect to 36 but for any integer b with $1 \leq b \leq 1296$, $b \to A$ never gives an MPTQ set.*

4. Construction of an infinite family of MPTQ sets

We shall discuss a technique using which we can generate infinitely many MPTQ sets starting with one.

Proposition 4.1. *Let A be an MPTQ set of positive integers. Let $p_1, p_2, \ldots, p_r$ be the distinct prime factors of the numbers in A. Then the set obtained from A by replacing each of the primes $p_1, p_2, \ldots, p_r$ respectively with r distinct pairwise coprime numbers $b_1, b_2, \ldots, b_r$ is also an MPTQ set.*

Proof. The proof follows from the fact that

$$\begin{aligned} b_1^{k_1} b_2^{k_2} \ldots b_r^{k_r} = b_1^{l_1} b_2^{l_2} \ldots b_r^{l_r} &\iff (k_1, k_2, \ldots, k_r) = (l_1, l_2, \ldots, l_r) \\ &\iff p_1^{k_1} p_2^{k_2} \ldots p_r^{k_r} = p_1^{l_1} p_2^{l_2} \ldots p_r^{l_r}. \end{aligned}$$ □

In particular, we may switch the primes $p_1, p_2, \ldots, p_r$ in A among themselves in order to get an MPTQ set.

Example 4.2. *$A = \{4, 9, 16, 18, 24, 36, 162, 216, 243, 432, 972\} \subset \{2^m 3^n : m, n \geq 0\}$ is an MPTQ set. By switching 2 and 3 in A, we get an MPTQ set $B = \{4, 9, 12, 32, 36, 48, 54, 81, 216, 288, 648\}$.*

Corollary 4.3. *Let A be an MPTQ set of positive integers. Let $p_1, p_2, \ldots, p_r$ be the distinct prime factors of the numbers in A. Then the set A^n obtained from A by replacing p_i with p_i^n for each $i, 1 \leq i \leq r$, is an MPTQ set.*

Example 4.4. *The set $A = \{4, 9, 16, 18, 24, 36, 162, 216, 243, 432, 972\}$ is an MPTQ, so is $A^2 = \{16, 81, 256, 324, 576, 1296, 26244, 46656, 59049, 186624, 944784\}$.*

Extending the above idea to rational numbers, we can replace $p_1, p_2, \ldots, p_r$ respectively by r distinct positive rational numbers $\frac{b_1}{d_1}, \frac{b_2}{d_2}, \ldots, \frac{b_r}{d_r}$, where $b_1, d_1, b_2, d_2, \ldots, b_r, d_r$ are all pairwise coprime, to generate an MPTQ set of rational numbers.

We shall mention the following lemmas for ease of reference.

Lemma 4.5. [3] *Let A be an MSTD set. Then for all $1 \neq r > 0$, $B = r^A$ is MPTQ.*

Lemma 4.6. [3] *Let A be an MPTQ set of positive numbers. Fix $1 \neq r > 0$. Then $B = \log_r A$ is MSTD.*

Corollary 4.3 shows that if A is MPTQ, then so is A^n. Our next result shows that the same is true even if we replace n by any nonzero real number and this is a consequence of the above two lemmas.

Proposition 4.7. *Let $r \in \mathbb{R} \setminus \{0\}$. A finite set A of positive real numbers is an MPTQ set if and only if A^r is an MPTQ set.*

Proof. For a given $r \in \mathbb{R} \setminus \{0\}$,

$$\begin{aligned} \text{a finite set } A \text{ is MPTQ} &\iff \log_2 A \text{ is MSTD} \\ &\iff r \log_2 A \text{ is MSTD} \end{aligned}$$

$$\begin{aligned} &\Longleftrightarrow \log_2 A^r \text{ is MSTD} \\ &\Longleftrightarrow A^r \text{ is MPTQ.} \end{aligned}$$

□

Using the above result one can generate infinitely many MPTQ sets starting with one. It also confirms the existence of MPTQ sets consisting of only perfect squares by taking $r = 2$.

The interesting fact is that no MSTD set with only perfect squares has been found in the literature. However, we can construct balanced sets of cardinality $2k$ with perfect squares for every natural number k.

A natural number n can be expressed as a sum of two perfect squares if it has the form,

$$n = 2^l p_1^{2a_1} p_2^{2a_2} \dots p_r^{2a_r} q_1^{b_1} q_2^{b_2} \dots q_s^{b_s},$$

where p_i and q_j are primes of the form $4k+3$ and $4k+1$ respectively, l, a_i and b_j are non-negative integers. If $B = (b_1+1)(b_2+1)\dots(b_s+1)$, then the number of representations of n as a sum of two perfect squares is,

$$r_0'(n) = \begin{cases} 0 & \text{if any } a_i \text{ is half-integer;} \\ \frac{1}{2}B & \text{if all } a_i \text{ are integers and } B \text{ is even;} \\ \frac{1}{2}(B-(-1)^l) & \text{if all } a_i \text{ are integers and } B \text{ is odd.} \end{cases}$$

Example 4.8. 2465 *is a number that can be written as a sum of two perfect squares in* 4 *different ways:*

$$\begin{aligned} 2465 &= 8^2 + 49^2 \\ &= 16^2 + 47^2 \\ &= 23^2 + 44^2 \\ &= 28^2 + 41^2. \end{aligned}$$

The set $A = \{8^2, 49^2, 16^2, 47^2, 23^2, 44^2, 28^2, 41^2\}$ *is balanced with* $|A+A| = 33 = |A-A|$.

Similarly using the summands from the expression of 8125, which is a sum of perfect squares in 5 different ways, we get the set $\{90^2, 5^2, 69^2, 58^2, 85^2, 30^2, 27^2, 86^2, 75^2, 50^2\}$, which is also balanced with 51 sums and differences.

More generally, let n be a number written as sum of 2 squares in k different ways:

$$n = a_1^2 + b_1^2 = a_2^2 + b_2^2 = \dots = a_k^2 + b_k^2.$$

Let $A = \{a_1^2, a_2^2, \dots, a_k^2, b_1^2, b_2^2, \dots, b_k^2\}$. Equality of two sums gives rise to four equal differences in $A - A$ and there are $(k-1) + \dots + 2 + 1$ such equalities in $A + A$, each of which yields four equal differences in $A - A$. Suppose, these are the only equalities among the sums in $A + A$, then we get the following:

$$\begin{aligned} |A+A| &= |A+A|_{max} - (k-1) \\ &= \frac{2k(2k+1)}{2} - (k-1) \\ &= 2k^2 + 1, \end{aligned}$$

and

$$\begin{aligned} |A - A| &= |A - A|_{max} - 4\frac{k(k-1)}{2} \\ &= 2k(2k-1) + 1 - 2k(k-1) \\ &= 2k^2 + 1, \end{aligned}$$

where $|A + A|_{max}$ and $|A - A|_{max}$ denote the the maximum possible cardinality of $A + A$ and $A - A$ respectively. Here $|A + A| = |A - A|$, so A is balanced. Thus, we can construct balanced sets of cardinality $2k$ with perfect squares for every natural number k using all the summands in the expression of a number that can be written as a sum of two perfect squares in k different ways.

The question: is there an MSTD set consisting of only perfect squares? can be restated as follows.

Problem 4.9. *For the binary quadratic forms $f(x, y) = x^2+y^2$ and $g(x, y) = x^2-y^2$ in variables x and y, determine whether there exist sets of integers A, B and C with $|C| \geq 2$ such that*

$$|f(A)| < |g(A)|,$$
$$|f(B)| > |g(B)|,$$
$$|f(C)| = |g(C)|.$$

The above problem is similar to the questions posed by M. Nathanson [14] and it has been completely solved for every pair of normalized binary linear forms $f(x, y) = u_1x + v_1y$ and $g(x, y) = u_2x + v_2y$ with integer coefficients [16].

Let P denote the set of all primes. If $A \subset P$ is MPTQ, then the removal of one of the elements of A would still give an MPTQ set. This is a contrapositive version of the Remark 3.3. Since the smallest cardinality of an MPTQ set of positive numbers is 8 (Theorem 1.4, [3]), we can conclude that an MPTQ set of primes does not exist. However, there exist MSTD sets consisting of primes. In [5], we find an MSTD set $\{19, 79, 109, 139, 229, 349, 379, 439\}$ of primes. As P does not contain any MPTQ set, we have the following corollary of Proposition 4.7.

Corollary 4.10. *For any $r \in \mathbb{R} \setminus \{0\}$, the set P^r does not contain any MPTQ set. In particular, a set consisting of n^{th} powers of primes is never an MPTQ.*

The above corollary holds good even if we replace P by a set consisting of positive integers that are pairwise coprime.

5. Geometric Progressions of Positive Real Numbers

For $n \in \mathbb{N}$ and $r \in \mathbb{R} \setminus \{0, \pm 1\}$, consider the geometric progression $G(r, n)$. Being symmetric with respect to r^{n-1}, $G(r, n)$ is balanced. Also, $b \to G(r, n)$ does not form an MPTQ set for any $b \in \mathbb{R} \setminus \{0\}$ as proved in [3]. We shall show that adjoining two distinct positive real numbers to $G(r, n)$, does not give an MPTQ in the case when $1 \neq r > 0$.

Proposition 5.1. *Let $n \in \mathbb{N}$, and $1 \neq r > 0$. For any two distinct positive real numbers x and y, the set $G(r, n)' = G(r, n) \cup \{x, y\}$ is not MPTQ.*

Proof. Consider the geometric progression $G(r, n)$ and two distinct positive real numbers x and y. Consider the set $\log_2 G(r, n)$ obtained by $2-$*log transformation* of $G(r, n)$. We note that $\log_2 G(r, n)$ forms an arithmetic progression with the common difference $\log_2 r$. Now the set $\log_2 G(r,n)'$ obtained by $\log_2 x,\ \log_2 y \to \log_2 G(r,n)$ is not an MSTD using the result proved in [2] that says an arithmetic progression in union with two arbitrary real numbers is not an MSTD. As $2-$ *log transformation* of an MPTQ set has to be an MSTD, it follows that $G(r,n)'$ is not an MPTQ. □

More generally, any finite set of positive numbers in a geometric progression in union with two distinct positive real numbers x and y is not an MPTQ. Further, it has been proved in [4] that the union of two arithmetic progressions with the same common difference is never an MSTD. Hence, the union of two geometric progressions of positive real numbers with the same common ratio can not be an MPTQ set.

6. MPTQ sets in $\mathbb{Z}_p^*$

Almost all previous research on MSTD sets focused exclusively on the integers, as opposed to other abelian groups. The first ever paper in which MSTD sets in finite abelian groups are considered is by Nathanson [15], who showed that families of MSTD sets of integers can be constructed from MSTD sets in finite abelian groups. Recent progress on MSTD sets in $\mathbb{Z}_n$ can be found in [17, 21]. We consider the multiplicative group $\mathbb{Z}_p^*$ for a prime p and imitate the idea of MPTQ sets.

For each prime p, the map $f : \mathbb{Z}_{p-1} \longrightarrow \mathbb{Z}_p^*$ given by $f(n) = a^n$, where a is a generator of $\mathbb{Z}_p^*$, is a group isomorphism. So if A is an MSTD in $\mathbb{Z}_{p-1}$ then $f(A)$ is an MPTQ in $\mathbb{Z}_p^*$ and vice versa. The group $\mathbb{Z}_p^*$ has exactly $\phi(p-1)$ generators so we have $\phi(p-1)$ isomorphisms from $\mathbb{Z}_{p-1}$ to $\mathbb{Z}_p^*$.

Example 6.1. *The set* $A = \{0,1,2,4,5,9\}$ *is MSTD in* $\mathbb{Z}_{12}$. *So the sets* $f(A)$, *under the above map for* $a = 2,6,7,11$ *respectively,* $\{1,2,3,4,5,6\}, \{1,2,5,6,9,10\}$, $\{1,7,8,9,10,11\}$ *and* $\{1,3,4,7,8,11\}$ *are all MPTQ sets* $\mathbb{Z}_{13}^*$.

Due to the above isomorphism, the number of MPTQ sets in $\mathbb{Z}_p^*$ is equal to the number of MSTD sets in $\mathbb{Z}_{p-1}$. The smallest additive cyclic group containing an MSTD set is $\mathbb{Z}_{12}$, so the smallest prime number p for which $\mathbb{Z}_p^*$ contains an MPTQ set is 13.

For any $A \subset \mathbb{Z}_p^*$ with $|A| > \dfrac{p}{2}$, $A.A = A/A = \mathbb{Z}_p^*$. So A is never an MPTQ.

Computations performed using Python Programming give the data presented in the following tables. Table 1 shows the number of MPTQ, MQTP, and balanced sets in $\mathbb{Z}_p^*$ for the first 9 primes p, and Table 2 represents the number of MPTQ sets of all possible cardinalities for all primes p upto 23.

Table 1. Number of MPTQ, MQTP and balanced sets in $\mathbb{Z}_p^*$ for small values of p

Prime p	$\|\mathbf{MPTQ}(\mathbb{Z}_\mathbf{p}^*)\|$	$\|\mathbf{MQTP}(\mathbb{Z}_\mathbf{p}^*)\|$	**No. of balanced sets**
$\mathbf{p=2,3,5}$	0	0	$2^{p-1}-1$
7	0	12	51
11	0	320	703
13	24	1380	2691
17	384	20288	44863
19	792	75702	185649
23	15224	943008	3236071

Table 2. Number of MPTQ sets in $\mathbb{Z}_p^*$ of different cardinalities for small values of p

Prime p	**n**	**No. of MPTQ sets of cardinality n**	$\|\mathbf{MPTQ}(\mathbb{Z}_\mathbf{p}^*)\|$
13	6	24	24
17	7	256	384
	8	128	
19	7	108	792
	8	432	
	9	252	
23	7	220	15224
	8	2640	
	9	6160	
	10	4840	
	11	1364	

For every prime $p \geq 13$, consider the following sets:

$$\begin{aligned} A_1 &= \{1, 2, \ldots, \frac{p-1}{2}\}, \\ A_2 &= \{\frac{p+1}{2}, \ldots, p-1\}, \\ A_3 &= \{2, 4, \ldots, p-1\}, \\ A_4 &= \{1, 3, \ldots, p-2\}. \end{aligned}$$

For each $i, 1 \leq i \leq 4$, $A_i.A_i = \mathbb{Z}_p^*$, and $A_i/A_i = \mathbb{Z}_p^* \setminus \{p-1\}$. So each A_i is an MPTQ. This confirms the existence of MSTD set of cardinality $(p-1)/2$ in $\mathbb{Z}_{p-1}$ for each prime p.

Lemma 6.2. *For each prime $p \geq 13$, $MSTD(\mathbb{Z}_{p-1})$ is nonempty.*

More generally, for each $n \in \mathbb{N}$, with $\phi(n) \geq 12$, of the form p^k or $2p^k$, where p is an odd prime and $k \in \mathbb{N}$, the additive group $\mathbb{Z}_{\phi(n)}$ is isomorphic to the multiplicative group $U(\mathbb{Z}_n^*)$. So the number of MSTD sets in $\mathbb{Z}_{\phi(n)}$ is equal to the number of MPTQ sets in $U(\mathbb{Z}_n^*)$. The set containing first $\phi(n)/2$ elements of $U(\mathbb{Z}_n^*)$ is an MPTQ in $U(\mathbb{Z}_n^*)$. Therefore $\mathbb{Z}_{\phi(n)}$ contains an MSTD set of cardinality $\phi(n)/2$.

Lemma 6.3. *For each $n \in \mathbb{N}$, with $\phi(n) \geq 12$, of the form p^k or $2p^k$, where p is an odd prime and $k \in \mathbb{N}$, $\mathbb{Z}_{\phi(n)}$ contains an MSTD set of cardinality $\phi(n)/2$.*

7. Conclusion

In Additive Number Theory, the study of MSTD sets has received great attention in the recent past. One can explore more about MSTD sets in $\mathbb{R}$ and $\mathbb{Z}_{p-1}$ using the notion of MPTQ sets in $\mathbb{R} \setminus \{0\}$ and $\mathbb{Z}_p^*$ respectively. In this regard, we have shown several ways of constructing an infinite family of MPTQ sets, a method of constructing balanced sets of perfect squares, and presented an open problem on binary quadratic forms. We have listed the number of MPTQ, MQTP, and balanced sets in $\mathbb{Z}_p^*$ for the first 9 prime numbers. Though we can establish a connection between MSTD sets and MPTQ sets via $r-$ *exponential transformation* and $r-$*log transformation*, these transformations do not preserve integers. So, there are still numerous non-trivial questions that can be explored regarding MPTQ sets. We conclude with some questions for future work:

- What is the minimum number of elements to be added to a symmetric set, in particular to a finite set of numbers in geometric progression, to form an MPTQ?
- What is the minimum cardinality of an MPTQ set of integers?
- Is there a set which is both MSTD and MPTQ modulo a prime number p?

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